a.) A set of elements must have two operations defined for its elements if it is to qualify as a vector space. What are they ? Describe them very briefly.

b.) Closure is required for both operations. Explain what is required by closure.

c.) Choose one of the two operations. In addition to closure, it satisfies four axioms. List them. At minimum provide the equation form of each axiom. If time permits briefly explain them.

a.) A certain set is a basis set for the vector space $\mathbb{V}$. What does that mean ?

b.) A set of n vectors $\{|V_1\rangle, |V_2\rangle, \ldots\ldots, |V_n\rangle\}$ that are elements of the vector space $\mathbb{V}$ are found to be linearly dependent, and it is found that the zero vector is not a member of this set. Starting with the definition for linear dependence, prove that at least one vector in the set other than $|V_1\rangle$, the first one listed, can be represented as a linear combination of the other vectors in the set.

c.) Show that the span of the set is unchanged if the linearly dependent vector found in part b.) is removed from the set.

Consider a linear mapping L of a vector space $\mathbb{V}$ into a space $\mathbb{W}$. Show that the image of $\mathbb{V}$ in $\mathbb{W}$ under the mapping is a vector space, a subspace of $\mathbb{W}$. Assume that $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field which is also the field for the linear map. Note that most of the properties of addition and multiplication are not issues. Why ? What are the issues that must be addressed ? For each v_i $\mathbb{V}$, $L(v_i)$ is the image of v_i. The image of $\mathbb{V}$ in $\mathbb{W}$ is the set of all such images. Linear: $L\,[\,a|v_1\rangle + b|v_2\rangle] = a\,L\,[\,|v_1\rangle] + b\,L\,[\,|v_2\rangle]$ for a,b $\mathbb{K}$

a.) Many quantities of physical interest are represented by real symmetric matrices. What does it mean to say a matrix is real symmetric ?

b.) Rotations of reference frames are represented by orthogonal matrices. What does it mean to say that a matrix is orthogonal ?

c.) The matrix $\mathbb{\lambda}$ is a 3 x 3 real orthogonal matrix. Use the properties of orthogonal matrices to compute $\sum_{i=1}^{3} \lambda_{2i}\, \lambda_{2i}$. Explain !

d.) Prove that the determinant of an orthogonal matrix is ± 1.

e.) Show that eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues ($\lambda_i \neq \lambda_j$) are orthogonal ($\langle v_j | v_i \rangle = \tilde{v}_j^t\, \tilde{v}_i = 0$).

Eigenvalue Eq^n: $\mathbb{M}\, \tilde{v}_i = \lambda_i\, \tilde{v}_i$ Transposed Eq^n: $\tilde{v}_i^t\, \mathbb{M}^{\mathbf{t}} = \tilde{v}_i^t\, \mathbb{M} = \lambda_i\, \tilde{v}_i^t$

One period of a function is given below.

$$f(t) = \begin{cases} 0 & for \quad -2 < t < -1 \\ -1 & for \quad -1 < t < 0 \\ 1 & for \quad 0 < t < 1 \\ 0 & for \quad 1 < t < 2 \end{cases}$$

a.) What is the period of the function ?

b.) What are the Fourier frequencies ω_p in its expansion ?

c.) Sketch the function for -4 < + 4.

d.) To what value does the Fourier series for *f(t)* converge for t = - 1 ?

e.) The Fourier coefficient b_p for this function is computed as:

$$b_p = \frac{2}{T}\int_{-T/2}^{T/2} \sin\left[\omega_p t\right]\ f(t)\ dt \qquad \underline{\textbf{\textit{show that}}}:\ b_p = \int_0^1 \sin\left[\frac{p\pi\ t}{2}\right]\ dt\,.$$

f.) Evaluate b_p for all p. Give forms for $p=4m,\ 4m+1,\ 4m+2,\ 4m+3$.

a.) Compute the divergence of $\vec{r} = r\,\hat{r}$ in Cartesian and spherical coordinates.

b.) Directly compute $\oint_{sphere\ surface} \vec{r}\ \hat{n}\, dA$ over the surface of a sphere of radius R centered on the origin.

c.) Directly compute $\int_{vol\ sphere} \vec{r}\ dV$ integrating over the volume of radius R centered on the origin.

a.) Compute $\vec{\nabla}\left(\frac{1}{r}\right)$ where $r = \sqrt{x^2 + y^2 + z^2}$ the distance from the origin.

b.) Compute $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right)$

c.) Compute $\vec{\nabla} \times \left(\frac{\vec{r}}{r^3}\right)$

d.) Compute the gradient of $\sin(k\,r)$ where k is a constant. $\vec{\nabla}\sin(k\,r) =$

a.) Give the expression for the line element $d\vec{r}$ in Cartesian, cylindrical and spherical coordinates.

b.) Sketch a coordinate 'cube' (a small volume element with pieces of the line element as three of its edges) in spherical coordinates that has one corner at (r, $\pi/4$, $\pi/4$).

c.) Give the expressions for the cylindrical coordinate directions $\hat{r}$ and $\hat{\theta}$ in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

d.) Compute $\dfrac{d\hat{r}}{d\theta}$.

a.) Display $\mathbb{M}^t$, given $\mathbb{M} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & -5 \end{bmatrix}$

b.) For $\mathbb{S} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}$ $\mathbb{T} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$ Compute $\mathbb{S}\ \mathbb{T}$.

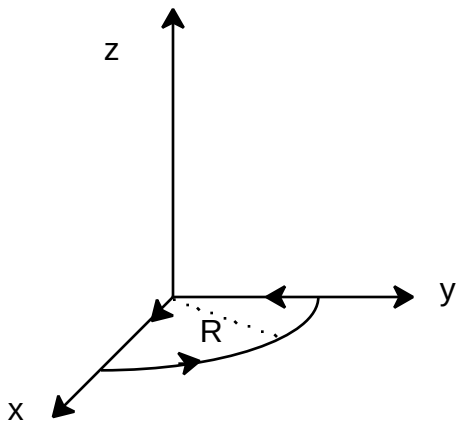
Given $|\ \mathbb{A}\ | = 4$ $|\ \mathbb{B}\ | = 2$ $|\ \mathbb{C}\ | = -1$

c.) $|\ \mathbb{A}\ \mathbb{B}\ \mathbb{C}\ | =$

d.) $|\ \mathbb{A}\ \mathbb{B}^{-1}\ | =$

e.) If $\mathbb{B}$ is a 2 x 2 matrix, $|\ 2\ \mathbb{B}\ | =$

f.) $|\ \mathbb{A}^t\ | =$



Spherical Coordinates

Compute: $\oint \vec{v} \quad d\vec{r}$ for the path shown.

It is the boundary of a quarter circle of radius R lying in the first quadrant of the *x-y* plane with its center at the origin. Data is given to be given in spherical coordinates. Work in spherical coordinates.

$\vec{v} = \phi \sin\theta \ \hat{r} - 4\theta\,\hat{\theta} + r\,\hat{\phi}$

Include the steps requested ;also use Stoke's thm.

a.) Answers in spherical coordinates. The path lies in a surface with θ = _____ ?

b.) By the RHR, what is the normal direction to the path ?

c.) Give $d\vec{r}$ *and* $\vec{v} \quad d\vec{r}$. Evaluate the inner product.

d.) Describe the path with values and/or ranges of values for each coordinate for each of the three segments into which you divide the path.

e.) Compute $\oint \vec{v} \quad d\vec{r}$ for the path.

f.) According to Stoke's theorem, the answer may be computed as _____ .

g.) Which component of $\vec{\ } \times \vec{v}$ is needed ?

h.) Evaluate the component of the curl needed to apply Stoke's theorem.

i.) Evaluate $\oint \vec{v} \quad d\vec{r}$ by computing a surface integral following.

The classic eigenvalue problem: $\mathbb{M}\,\vec{v} = \lambda\,\mathbb{1}\,\vec{v} = \lambda\,\vec{v}$

Given $\mathbb{M} = \begin{bmatrix} 100 & 36 & 0 \\ 36 & 79 & 0 \\ 0 & 0 & 120 \end{bmatrix}$, find the eigenvalues and eigenvectors.

A rotation of coordinates by θ about the x-axis is represented as

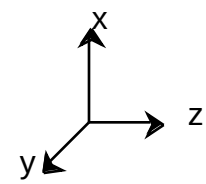
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \text{ and } \varphi \text{ about the z-axis as } \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

in general
$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} (\hat{i}' \cdot \hat{i}) & (\hat{i}' \cdot \hat{j}) & (\hat{i}' \cdot \hat{k}) \\ (\hat{j}' \cdot \hat{i}) & (\hat{j}' \cdot \hat{j}) & (\hat{j}' \cdot \hat{k}) \\ (\hat{k}' \cdot \hat{i}) & (\hat{k}' \cdot \hat{j}) & (\hat{k}' \cdot \hat{k}) \end{bmatrix}$$

a.) Give the matrix that represents a rotation of 45° about the z axis followed by 45^{0} about the (then) x-axis.

b.) A point had coordinates in the unprimed frame of (0,2,1). What are its coordinates in the rotated frame ?

c.) For the axis set shown, the *y*-axis is 'out of the page'. Imagine rotating the axis by $+30^{o}$ about the *y*-axis. Sketch the *x' and z' axes*. Give the form of the matrix for a rotation by $+\psi$ about the *y*-axis.



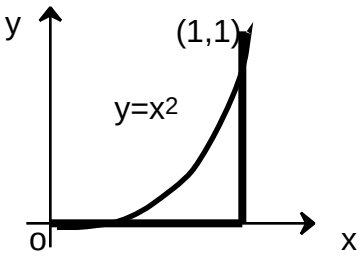
coordinate triad

NEW PART: The goal is to rotate the current z-axis to lie along the line x=y=z in the current system.

d.) Draw a triad with *z* up and *x* out of the page. Rotate by 45^{0} about the *z* axis. Sketch the *x'y'z'* triad on the figure. Give the matrix for the transformation.

e.) Draw the *x'y'z'* triad on a new figure. Rotate by ψ about the *y'* axis to leave the *z"* axis along the line *x=y=z* as seen in the original *xyz* frame. Give the matrix for the transformation ($\cos\psi = 1/\sqrt{3}$). What is $\sin\psi$?

$$\int_{path} \vec{F} \; d\vec{r} \; ; \quad \vec{F} = F_x \; \hat{i} + F_y \; \hat{j} \qquad \vec{F} = x\,y\,\hat{i} - y^2 \; \hat{j}$$



path 1: *(0,0) to (1,1) along* $y = x^2$

path 2:

straight (0,0) to (1,1) plus straight (1,0) to (1,1)

a.) Compute the path integral of the vector field given above for path one.

b.) Compute the path integral of the vector field given above for path two.

c.) Compute $\vec{\ } \times \vec{F}$.

Consider the matrix $\mathbb{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

Show all your work. You may use row reduction, methods involving cofactors or other expansion by hand. Correct answers without details are worth only 30%. Don't just ask your TI-92.

a.) Compute the determinant of $\mathbb{B} = |\mathbb{B}|$.

b.) Compute the inverse $\mathbb{B}^{-1}$.

a.) Show that $(\mathbb{A}\ \mathbb{B})^t = \mathbb{B}^t\ \mathbb{A}^t$. Start with $(\mathbb{A}\ \mathbb{B})_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$. Express the elements of the transpose, elements in the sum in terms of the elements of the transposed matrices and reassemble using the definition of the matrix product.

b.) Give a representation for $(\mathbb{A}\ \mathbb{B})^{-1}$ in terms of $\mathbb{A}^{-1}$ and $\mathbb{B}^{-1}$. Show that $\mathbb{A}\ \mathbb{B}$ times your proposed answer equals $\mathbb{1}$.

a.) Electromagnetic theory states that $\vec{B}(\vec{r}) = \vec{\ } \times \vec{A}(\vec{r})$. Given $\vec{A}(\vec{r}) = \frac{1}{2}\,\vec{B}_0 \times \vec{r}$ where $\vec{B}_0$ is a constant vector, find $\vec{B}(\vec{r})$. Use the product rules. Include explicit evaluations of $\left(\vec{B}_o\ \vec{\ }\right)$, $\left(\vec{B}_o\ \vec{\ }\right)\vec{r}$ and $\vec{\ }\ \vec{r}$ in the Cartesian representation. Reassemble the pieces into a vector statement of the result.

b.) The force on an electric dipole in a non-uniform electric field can be computed as $\vec{F}_{dipole} = \left[\vec{p}\ \vec{\ }\right]\vec{E}$. Consider the dipole in the field of a point charge q located at the origin $\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$ and the dipole is a distance r away and is directed radially away. $\vec{p} = p\hat{r}$ Evaluate $\left[\vec{p}\ \vec{\ }\right]$ carefully in spherical coordinates. Don't forget the inner products in the brackets.

c.) Repeat b for the dipole directed along a line of longitude. $\vec{p} = p\hat{\theta}$

SPHERICAL COORDINATES	
$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$	$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\ \hat{\phi}$
$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$	$\frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\ \hat{\phi}$
$\frac{\partial\hat{\phi}}{\partial\theta} = 0$	$\frac{\partial\hat{\phi}}{\partial\phi} = -\sin\theta\ \hat{r} - \cos\theta\ \hat{\theta}$

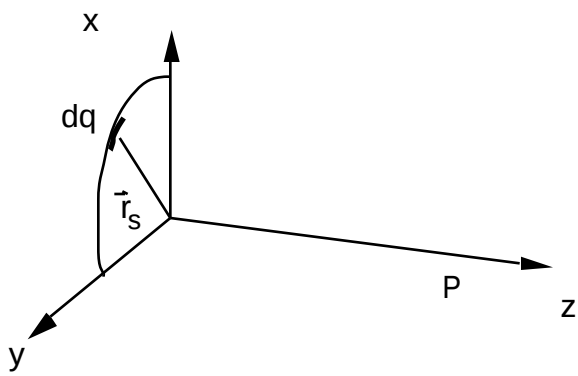
Compute the electrostatic field due to one quarter of a charged circular loop of radius R centered on the origin in the x-y plane at points on the z-axis. Assume a uniform linear charge density λ.

$$\vec{E}(\vec{r}_p) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{all\\sources}} \frac{\left(\vec{r}_p - \vec{r}_s\right) dq_s}{\left|\vec{r}_p - \vec{r}_s\right|^3}; \quad k_{Coulomb} = \frac{1}{4\pi\varepsilon_0}; \quad V(\vec{r}_p) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{all\\sources}} \frac{dq_s}{\left|\vec{r}_p - \vec{r}_s\right|}$$

Electric field Coulomb's k Potential

$$dq_s = \begin{cases} \lambda\, d\ell & \textit{charge spread along a line} \\ \sigma\, dA & \textit{charge spread over an area} \\ \rho\, dV & \textit{charge spread throughout a volume} \end{cases}$$

a.) What are your choices for integration variables ?



Very Rough Sketch !

b.) Give an expression for $\vec{r}_s$

$\vec{r}_s =$

c.) $\vec{r}_p =$

d.) $\vec{r}_{sp} =$

e.) $\hat{r}_{sp} =$

f.) Give the limits of integration for each integration variable.

g.) Give the expression for the element of length $d\ell$ along the arc.

h.) Give an expression for dq in terms of your integration variable.

i.) Evaluate the integral to find the field at P due to the quarter circular arc.

(75%) Given $G(\vec{r})$ is a scalar valued function with continuous derivatives through second order. Use the Cartesian representation to show that: $\vec{\ } \times \vec{\ } G = 0$. This result is summarized in the operator relation $\vec{\ } \times \vec{\ } = 0$.

Give an explicit prose justification for the equality of terms declared equal.

(25%) Compute $\left[\vec{\ } \times \vec{\ } G\right]_r$ in spherical coordinates. Use the forms on the identities and that $G(\vec{r})$ is a scalar valued function with continuous derivatives through second order. Note: Only the radial component need by computed.